2.1 Transformations of Quadratic Functions

COMMON CORE

Learning Standards
HSF-IF.C.7c
HSF-BF.B.3

Essential Question

How do the constants a, h, and k affect the graph of the quadratic function $g(x) = a(x - h)^2 + k$?

The parent function of the quadratic family is $f(x) = x^2$. A transformation of the graph of the parent function is represented by the function $g(x) = a(x - h)^2 + k$, where $a \neq 0$.

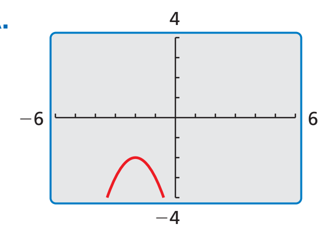
EXPLORATION 1 Identifying Graphs of Quadratic Functions

Work with a partner. Match each quadratic function with its graph. Explain your reasoning. Then use a graphing calculator to verify that your answer is correct.

a. $g(x) = -(x - 2)^2$ **b.** $g(x) = (x - 2)^2 + 2$ **c.** $g(x) = -(x + 2)^2 - 2$

d. $g(x) = 0.5(x - 2)^2 - 2$ **e.** $g(x) = 2(x - 2)^2$ **f.** $g(x) = -(x + 2)^2 + 2$

A.



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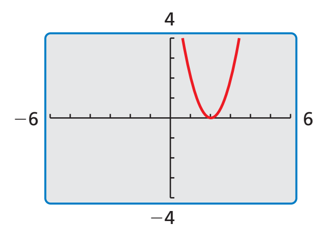

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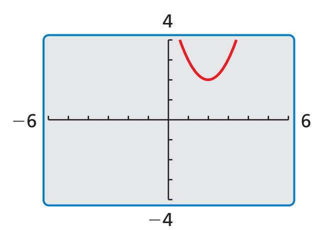

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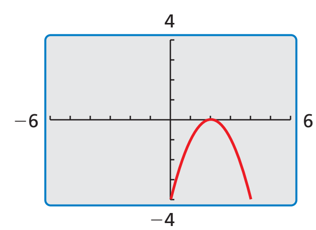

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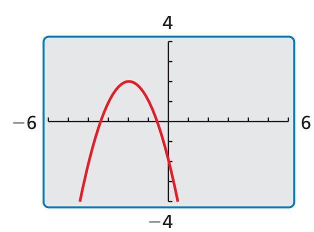

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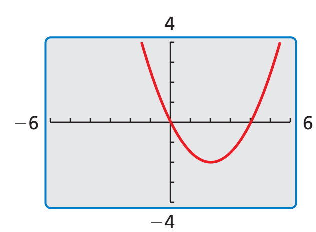


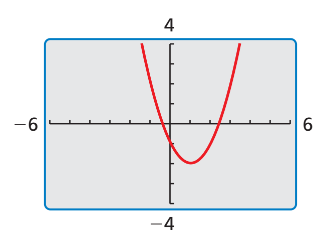
LOOKING FOR STRUCTURE

To be proficient in math, you need to look closely to discern a pattern or structure.

Communicate Your Answer

2. How do the constants a, h, and k affect the graph of the quadratic function $g(x) = a(x - h)^2 + k$?

3. Write the equation of the quadratic function whose graph is shown at the right. Explain your reasoning. Then use a graphing calculator to verify that your equation is correct.



2.1 Lesson

Core Vocabulary

quadratic function, *p. 48*
parabola, *p. 48*
vertex of a parabola, *p. 50*
vertex form, *p. 50*

Previous
transformations

What You Will Learn

- Describe transformations of quadratic functions.
- Write transformations of quadratic functions.

Describing Transformations of Quadratic Functions

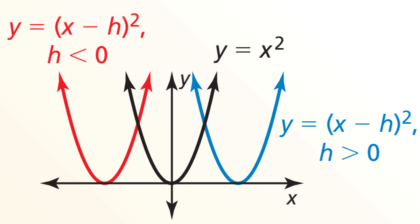
A **quadratic function** is a function that can be written in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The U-shaped graph of a quadratic function is called a **parabola**.

In Section 1.1, you graphed quadratic functions using tables of values. You can also graph quadratic functions by applying transformations to the graph of the parent function $f(x) = x^2$.

Core Concept

Horizontal Translations

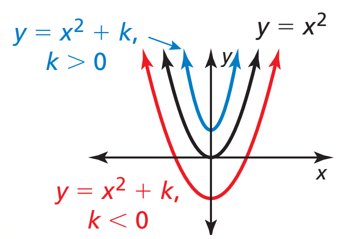
$$f(x) = x^2$$
$$f(x - h) = (x - h)^2$$



- shifts left when $h < 0$
- shifts right when $h > 0$

Vertical Translations

$$f(x) = x^2$$
$$f(x) + k = x^2 + k$$



- shifts down when $k < 0$
- shifts up when $k > 0$

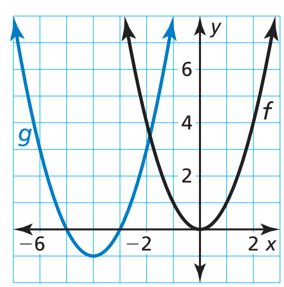
EXAMPLE 1 Translations of a Quadratic Function

Describe the transformation of $f(x) = x^2$ represented by $g(x) = (x + 4)^2 - 1$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = (x - h)^2 + k$. Rewrite the function to identify h and k.

$$g(x) = (x - \underbrace{(-4)}_{h})^2 + \underbrace{(-1)}_{k}$$



- Because $h = -4$ and $k = -1$, the graph of g is a translation 4 units left and 1 unit down of the graph of f.

Monitoring Progress

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Describe the transformation of $f(x) = x^2$ represented by g. Then graph each function.

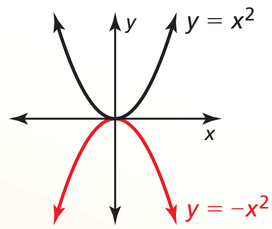
1. $g(x) = (x - 3)^2$ **2.** $g(x) = (x - 2)^2 - 2$ **3.** $g(x) = (x + 5)^2 + 1$

Core Concept

Reflections in the x-Axis

$f(x) = x^2$

$-f(x) = -(x^2) = -x^2$

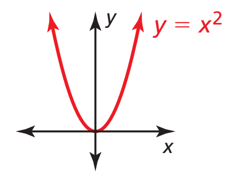


flips over the x-axis

Reflections in the y-Axis

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

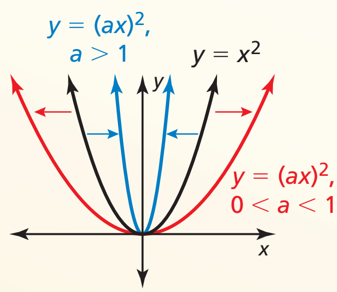


$y = x^2$ is its own reflection in the y-axis.

Horizontal Stretches and Shrinks

$f(x) = x^2$

$f(ax) = (ax)^2$

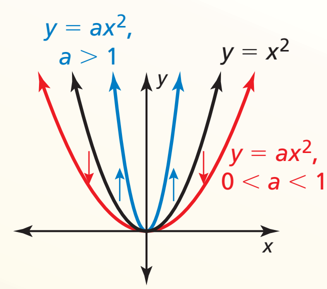


- horizontal stretch (away from y-axis) when $0 < a < 1$
- horizontal shrink (toward y-axis) when $a > 1$

Vertical Stretches and Shrinks

$f(x) = x^2$

$a \cdot f(x) = ax^2$



- vertical stretch (away from x-axis) when $a > 1$
- vertical shrink (toward x-axis) when $0 < a < 1$

EXAMPLE 2 Transformations of Quadratic Functions

Describe the transformation of $f(x) = x^2$ represented by g. Then graph each function.

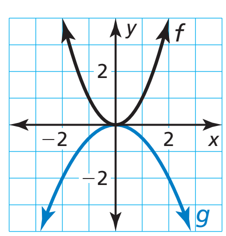
a. $g(x) = -\frac{1}{2}x^2$

b. $g(x) = (2x)^2 + 1$

SOLUTION

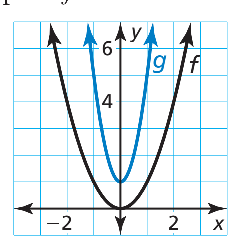
a. Notice that the function is of the form $g(x) = -ax^2$, where $a = \frac{1}{2}$.

So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{2}$ of the graph of f.



b. Notice that the function is of the form $g(x) = (ax)^2 + k$, where $a = 2$ and $k = 1$.

So, the graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ followed by a translation 1 unit up of the graph of f.



LOOKING FOR STRUCTURE

In Example 2b, notice that $g(x) = 4x^2 + 1$. So, you can also describe the graph of g as a vertical stretch by a factor of 4 followed by a translation 1 unit up of the graph of f.

Monitoring Progress

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Describe the transformation of $f(x) = x^2$ represented by g. Then graph each function.

4. $g(x) = \left(\frac{1}{3}x\right)^2$ **5.** $g(x) = 3(x - 1)^2$ **6.** $g(x) = -(x + 3)^2 + 2$

Writing Transformations of Quadratic Functions

The lowest point on a parabola that opens up or the highest point on a parabola that opens down is the **vertex**. The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$ and the vertex is (h, k).

$$f(x) = a(x - h)^2 + k$$

a indicates a reflection in the *x*-axis and/or a vertical stretch or shrink.

h indicates a horizontal translation.

k indicates a vertical translation.

EXAMPLE 3 Writing a Transformed Quadratic Function

Let the graph of g be a vertical stretch by a factor of 2 and a reflection in the x-axis, followed by a translation 3 units down of the graph of $f(x) = x^2$. Write a rule for g and identify the vertex.

SOLUTION

Method 1 Identify how the transformations affect the constants in vertex form.

$$\left.\begin{array}{l}\text{reflection in } x\text{-axis}\\ \text{vertical stretch by } 2\end{array}\right\} a = -2$$

$$\text{translation 3 units down}\} \; k = -3$$

Write the transformed function.

$g(x) = a(x - h)^2 + k$ Vertex form of a quadratic function

$= -2(x - 0)^2 + (-3)$ Substitute -2 for a, 0 for h, and -3 for k.

$= -2x^2 - 3$ Simplify.

▶ The transformed function is $g(x) = -2x^2 - 3$. The vertex is $(0, -3)$.

Method 2 Begin with the parent function and apply the transformations one at a time in the stated order.

First write a function h that represents the reflection and vertical stretch of f.

$h(x) = -2 \cdot f(x)$ Multiply the output by -2.

$= -2x^2$ Substitute x^2 for $f(x)$.

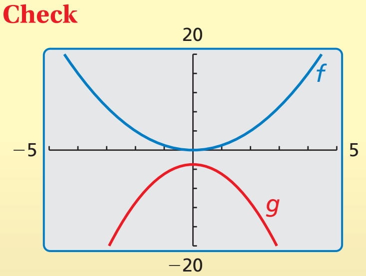
Then write a function g that represents the translation of h.

$g(x) = h(x) - 3$ Subtract 3 from the output.

$= -2x^2 - 3$ Substitute $-2x^2$ for $h(x)$.

▶ The transformed function is $g(x) = -2x^2 - 3$. The vertex is $(0, -3)$.

Check



EXAMPLE 4 Writing a Transformed Quadratic Function

Let the graph of g be a translation 3 units right and 2 units up, followed by a reflection in the y-axis of the graph of $f(x) = x^2 - 5x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the translation of f.

$h(x) = f(x - 3) + 2$	Subtract 3 from the input. Add 2 to the output.
$= (x - 3)^2 - 5(x - 3) + 2$	Replace x with $x - 3$ in $f(x)$.
$= x^2 - 11x + 26$	Simplify.

Step 2 Then write a function g that represents the reflection of h.

$g(x) = h(-x)$	Multiply the input by -1.
$= (-x)^2 - 11(-x) + 26$	Replace x with $-x$ in $h(x)$.
$= x^2 + 11x + 26$	Simplify.

REMEMBER

To multiply two binomials, use the FOIL Method.

First Inner

$(x + 1)(x + 2) = x^2 + 2x + x + 2$

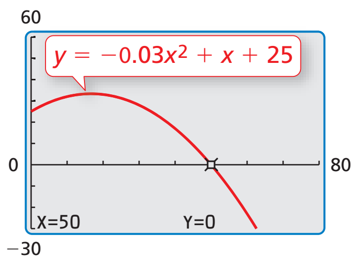
Outer Last

EXAMPLE 5 Modeling with Mathematics

The height h (in feet) of water spraying from a fire hose can be modeled by $h(x) = -0.03x^2 + x + 25$, where x is the horizontal distance (in feet) from the fire truck. The crew raises the ladder so that the water hits the ground 10 feet farther from the fire truck. Write a function that models the new path of the water.

SOLUTION

1. **Understand the Problem** You are given a function that represents the path of water spraying from a fire hose. You are asked to write a function that represents the path of the water after the crew raises the ladder.
2. **Make a Plan** Analyze the graph of the function to determine the translation of the ladder that causes water to travel 10 feet farther. Then write the function.
3. **Solve the Problem** Graph the transformed function.



Because $h(50) = 0$, the water originally hits the ground 50 feet from the fire truck. The range of the function in this context does not include negative values. However, by observing that $h(60) = -23$, you can determine that a translation 23 units (feet) up causes the water to travel 10 feet farther from the fire truck.

$g(x) = h(x) + 23$	Add 23 to the output.
$= -0.03x^2 + x + 48$	Substitute for $h(x)$ and simplify.

▶ The new path of the water can be modeled by $g(x) = -0.03x^2 + x + 48$.

4. **Look Back** To check that your solution is correct, verify that $g(60) = 0$.

$$g(60) = -0.03(60)^2 + 60 + 48 = -108 + 60 + 48 = 0 \checkmark$$

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7. Let the graph of g be a vertical shrink by a factor of $\frac{1}{2}$ followed by a translation 2 units up of the graph of $f(x) = x^2$. Write a rule for g and identify the vertex.
8. Let the graph of g be a translation 4 units left followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of $f(x) = x^2 + x$. Write a rule for g.
9. **WHAT IF?** In Example 5, the water hits the ground 10 feet closer to the fire truck after lowering the ladder. Write a function that models the new path of the water.

2.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The graph of a quadratic function is called a(n) ________.

2. **VOCABULARY** Identify the vertex of the parabola given by $f(x) = (x + 2)^2 - 4$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, describe the transformation of $f(x) = x^2$ represented by g. Then graph each function. *(See Example 1.)*

3. $g(x) = x^2 - 3$

4. $g(x) = x^2 + 1$

5. $g(x) = (x + 2)^2$

6. $g(x) = (x - 4)^2$

7. $g(x) = (x - 1)^2$

8. $g(x) = (x + 3)^2$

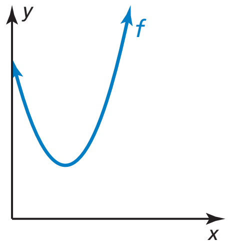
9. $g(x) = (x + 6)^2 - 2$

10. $g(x) = (x - 9)^2 + 5$

11. $g(x) = (x - 7)^2 + 1$

12. $g(x) = (x + 10)^2 - 3$

ANALYZING RELATIONSHIPS **In Exercises 13–16, match the function with the correct transformation of the graph of f. Explain your reasoning.**



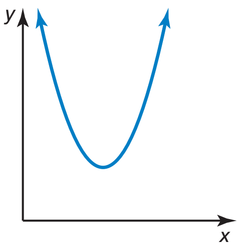
13. $y = f(x - 1)$

14. $y = f(x) + 1$

15. $y = f(x - 1) + 1$

16. $y = f(x + 1) - 1$

A.



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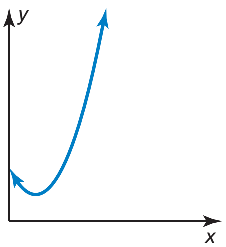

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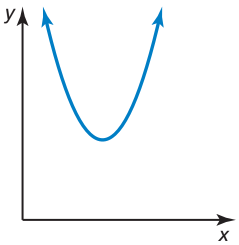

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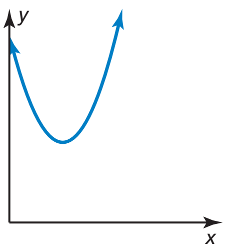


In Exercises 17–24, describe the transformation of $f(x) = x^2$ represented by g. Then graph each function. *(See Example 2.)*

17. $g(x) = -x^2$

18. $g(x) = (-x)^2$

19. $g(x) = 3x^2$

20. $g(x) = \frac{1}{3}x^2$

21. $g(x) = (2x)^2$

22. $g(x) = -(2x)^2$

23. $g(x) = \frac{1}{5}x^2 - 4$

24. $g(x) = \frac{1}{2}(x - 1)^2$

ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in analyzing the graph of $f(x) = -6x^2 + 4$.**

25. ✗ The graph is a reflection in the *y*-axis and a vertical stretch by a factor of 6, followed by a translation 4 units up of the graph of the parent quadratic function.

26. ✗ The graph is a translation 4 units down, followed by a vertical stretch by a factor of 6 and a reflection in the *x*-axis of the graph of the parent quadratic function.

USING STRUCTURE **In Exercises 27–30, describe the transformation of the graph of the parent quadratic function. Then identify the vertex.**

27. $f(x) = 3(x + 2)^2 + 1$

28. $f(x) = -4(x + 1)^2 - 5$

29. $f(x) = -2x^2 + 5$

30. $f(x) = \frac{1}{2}(x - 1)^2$

In Exercises 31–34, write a rule for g described by the transformations of the graph of f. Then identify the vertex. *(See Examples 3 and 4.)*

31. $f(x) = x^2$; vertical stretch by a factor of 4 and a reflection in the x-axis, followed by a translation 2 units up

32. $f(x) = x^2$; vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the y-axis, followed by a translation 3 units right

33. $f(x) = 8x^2 - 6$; horizontal stretch by a factor of 2 and a translation 2 units up, followed by a reflection in the y-axis

34. $f(x) = (x + 6)^2 + 3$; horizontal shrink by a factor of $\frac{1}{2}$ and a translation 1 unit down, followed by a reflection in the x-axis

USING TOOLS **In Exercises 35–40, match the function with its graph. Explain your reasoning.**

35. $g(x) = 2(x - 1)^2 - 2$

36. $g(x) = \frac{1}{2}(x + 1)^2 - 2$

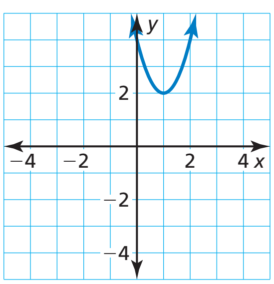
37. $g(x) = -2(x - 1)^2 + 2$

38. $g(x) = 2(x + 1)^2 + 2$

39. $g(x) = -2(x + 1)^2 - 2$

40. $g(x) = 2(x - 1)^2 + 2$

A.



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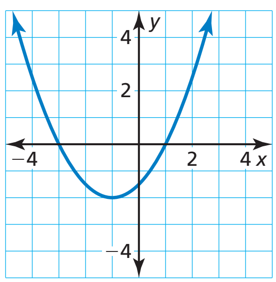

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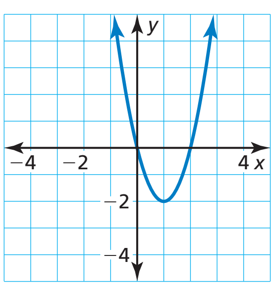

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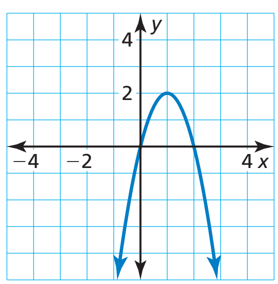

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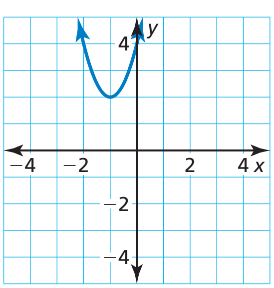

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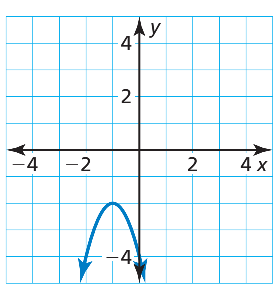


JUSTIFYING STEPS **In Exercises 41 and 42, justify each step in writing a function g based on the transformations of $f(x) = 2x^2 + 6x$.**

41. translation 6 units down followed by a reflection in the x-axis

$$\begin{aligned} h(x) &= f(x) - 6 \\ &= 2x^2 + 6x - 6 \\ g(x) &= -h(x) \\ &= -(2x^2 + 6x - 6) \\ &= -2x^2 - 6x + 6 \end{aligned}$$

42. reflection in the y-axis followed by a translation 4 units right

$$\begin{aligned} h(x) &= f(-x) \\ &= 2(-x)^2 + 6(-x) \\ &= 2(x)^2 - 6x \\ g(x) &= h(x - 4) \\ &= 2(x - 4)^2 + 6(x - 4) \\ &= 2x^2 - 10x + 8 \end{aligned}$$

43. **MODELING WITH MATHEMATICS** The function $h(x) = -0.03(x - 14)^2 + 6$ models the jump of a red kangaroo, where x is the horizontal distance traveled (in feet) and $h(x)$ is the height (in feet). When the kangaroo jumps from a higher location, it lands 5 feet farther away. Write a function that models the second jump. *(See Example 5.)*

44. **MODELING WITH MATHEMATICS** The function $f(t) = -16t^2 + 10$ models the height (in feet) of an object t seconds after it is dropped from a height of 10 feet on Earth. The same object dropped from the same height on the moon is modeled by $g(t) = -\frac{8}{3}t^2 + 10$. Describe the transformation of the graph of f to obtain g. From what height must the object be dropped on the moon so it hits the ground at the same time as on Earth?

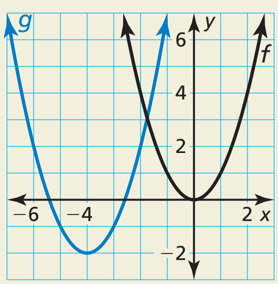
45. MODELING WITH MATHEMATICS Flying fish use their pectoral fins like airplane wings to glide through the air.

a. Write an equation of the form $y = a(x - h)^2 + k$ with vertex (33, 5) that models the flight path, assuming the fish leaves the water at (0, 0).

b. What are the domain and range of the function? What do they represent in this situation?

c. Does the value of a change when the flight path has vertex (30, 4)? Justify your answer.

46. HOW DO YOU SEE IT? Describe the graph of g as a transformation of the graph of $f(x) = x^2$.



47. COMPARING METHODS Let the graph of g be a translation 3 units up and 1 unit right followed by a vertical stretch by a factor of 2 of the graph of $f(x) = x^2$.

a. Identify the values of a, h, and k and use vertex form to write the transformed function.

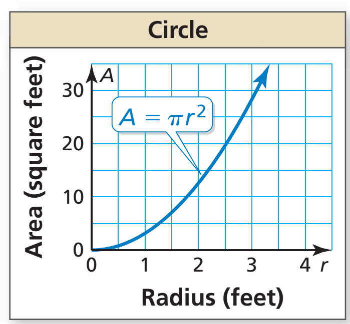
b. Use function notation to write the transformed function. Compare this function with your function in part (a).

c. Suppose the vertical stretch was performed first, followed by the translations. Repeat parts (a) and (b).

d. Which method do you prefer when writing a transformed function? Explain.

48. THOUGHT PROVOKING A jump on a pogo stick with a conventional spring can be modeled by $f(x) = -0.5(x - 6)^2 + 18$, where x is the horizontal distance (in inches) and $f(x)$ is the vertical distance (in inches). Write at least one transformation of the function and provide a possible reason for your transformation.

49. MATHEMATICAL CONNECTIONS The area of a circle depends on the radius, as shown in the graph. Describe two different transformations of the graph that model the area of the circle if the area is doubled.

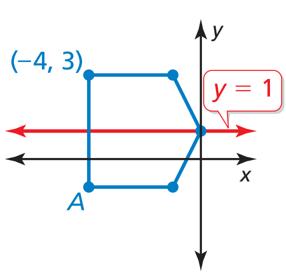


Maintaining Mathematical Proficiency

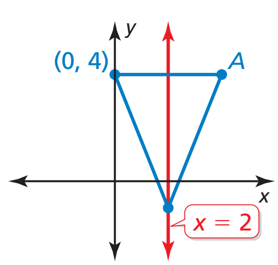
Reviewing what you learned in previous grades and lessons

A line of symmetry for the figure is shown in red. Find the coordinates of point A.
(Skills Review Handbook)

50.



51.



52.